MATH 347: FUNDAMENTAL MATHEMATICS, FALL 2015

PRACTICE PROBLEMS FOR MITERM 1

1. Prove that for all sets A, B

(a) $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$.

Solution 1. For an arbitrary element x,

$$\begin{aligned} x \in A \cup B &\iff (x \in A) \vee (x \in B) \\ &\iff (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in B) \vee (x \in B \wedge x \notin A) \\ &\iff (x \in A - B) \vee (x \in A \cap B) \vee (x \in B - A) \\ &\iff x \in (A - B) \cup (A \cap B) \cup (B - A). \end{aligned}$$

□

Solution 2 (long, but still correct). We show that one is a subset of the other and vice versa.

$A \cup B \subseteq (A - B) \cup (A \cap B) \cup (B - A)$: Fix arbitrary $x \in A \cup B$. Only the following cases are possible and we handle each case separately.

Case 1: $x \in A$ but $x \notin B$. Then $x \in A - B$ and thus $x \in (A - B) \cup (A \cap B) \cup (B - A)$.

Case 2: $x \in A$ and $x \in B$. Then $x \in A \cap B$ and thus $x \in (A - B) \cup (A \cap B) \cup (B - A)$.

Case 3: $x \in B$ but $x \notin A$. Then $x \in B - A$ and thus $x \in (A - B) \cup (A \cap B) \cup (B - A)$.

$(A - B) \cup (A \cap B) \cup (B - A) \subseteq A \cup B$: Fix arbitrary $x \in (A - B) \cup (A \cap B) \cup (B - A)$. If $x \in (A - B) \cup (A \cap B)$, then, in particular, $x \in A$, so $x \in A \cup B$ and we are done. If $x \in B - A$, then, in particular, $x \in B$, so $x \in A \cup B$ and we are done. □

(b) $(A \cup B) - B = A - B$.

Solution. We show that one is a subset of the other and vice versa.

$A - B \subseteq (A \cup B) - B$: Fix arbitrary $x \in A - B$, so $x \in A$ and $x \notin B$. Because $x \in A$, we also have $x \in A \cup B$, so by the definition of $-$, we get that $x \in (A \cup B) - B$.

$(A \cup B) - B \subseteq A - B$: Fix arbitrary $x \in (A \cup B) - B$, so $x \in A \cup B$ and $x \notin B$. $x \in A \cup B$ means that $x \in A$ or $x \in B$, but we have that $x \notin B$, so it must be that $x \in A$. Thus, $x \in A - B$. □

2. Prove or give a counter-example:

(a) For every function $f : X \to Y$ and every $B \subseteq Y$, $I_f(B)^c = I_f(B^c)$.

Solution. We prove this. Before we even start, we recall the definition of $I_f(D)$ for a subset $D \subseteq Y$:

$$I_f(D) := \{x \in X : f(x) \in D\}.$$

Now we are ready to start the proof. Fixing arbitrary $x \in X$, we show that $x \in I_f(B)^c \Leftrightarrow x \in I_f(B^c)$. Indeed,

$$\begin{aligned} x \in I_f(B)^c &\iff x \notin I_f(B) \\ &\iff f(x) \notin B \\ &\iff f(x) \in B^c \\ &\iff x \in I_f(B^c). \end{aligned}$$

□

(b) For every function $f : X \to Y$ and every $A \subseteq X$, $f(A)^c = f(A^c)$.

Solution. This isn't true in general and here is a counter-example. Let $X = \{1, 2\}, Y = \{3\}$ and define $f : X \to Y$ by $f(1) := 3$, $f(2) := 3$. Take $A = \{1\}$. Then $f(A) = \{3\}$, so $f(A)^c = \varnothing$. However, $A^c = \{2\}$, so $f(A^c) = \{3\}$. Thus, in this example, $f(A)^c$ is a strict subset of $f(A^c)$. □

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows: for $x \in \mathbb{R}$,

$$f(x) = \begin{cases} x^2 & \text{if } x \le -1 \\ |x| & \text{if } -1 \le x \le 1 \\ x & x \ge 0 \end{cases}.$$

Is f a well-defined function? Justify your answer.

Solution. To check that the function is well-defined, we need to check that in the overlapping cases, the values are the same.

The first overlap is $x = -1$: by the first line it is $(-1)^2 = 1$ and by the second line it is $|-1| = 1$, so they are equal, and thus, f is well-defined at -1.

Now the second overlap is when $x \in [0, 1]$. According to the second line, $f(x) = |x|$, but since $x \ge 0$, $|x| = x$, so $f(x) = x$, which coincides with the third line, so f is well-defined on $[0, 1]$. □

4. (a) Let $g : \mathbb{R} \to \mathbb{R}$ be the absolute value function, i.e. $g(x) = |x|$ for each $x \in \mathbb{R}$. What is $I_g\big(g([-1, 0))\big)$?

Solution. $g\big([-1, 0)\big) = (0, 1]$ and $I_g\big((0, 1]\big) = [-1, 0) \cup (0, 1]$. □

(b) In general, for an arbitrary function $f : X \to Y$ and $A \subseteq X$, what is the relation between A and $I_f\big(f(A)\big)$? Prove your answer.

Solution. The relation is $A \subseteq I_f\big(f(A)\big)$. To prove it, fix arbitrary $x \in A$. Thus, $f(x) \in f(A)$, so by the definition of $I_f\big(f(A)\big)$ (recalled above), $x \in I_f\big(f(A)\big)$. □

5. Recall the definition of linear independence for points (vectors) in $\mathbb{R}^n$.

Definition. Vectors $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_k \in \mathbb{R}^n$ are called *linearly independent* if

$$\forall a_1, a_2, \ldots, a_k \in \mathbb{R}\Big[a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_k\vec{v}_k = \vec{0} \implies \big(\forall i \le k, a_i = 0\big)\Big].$$

Vectors $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_k \in \mathbb{R}^n$ are said to be *linearly dependent* if they are not linearly independent.

(a) Write out explicitly what it means for vectors $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_k \in \mathbb{R}^n$ to be linearly dependent. The only negation sign/word in your sentence should be negating equality $\neq$.

Solution. $\exists a_1, a_2, \ldots, a_k \in \mathbb{R}\Big[a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_k\vec{v}_k = \vec{0} \wedge \big(\exists i \le k, a_i \neq 0\big)\Big]$. □

(b) Are the vectors $(1,1)$ and $(1,0)$ linearly independent? Prove your answer.

Solution. Yes, they are. To prove it, we fix arbitrary $a_1, a_2 \in \mathbb{R}$ and suppose the hypothesis of the above implication holds, namely:

$$a_1(1,1) + a_2(1,0) = (0,0).$$

We need to show that $a_1 = 0$ and $a_2 = 0$. Simplifying the left-hand side of the above equation, we get:

$$a_1(1,1) + a_2(1,0) = (a_1, a_1) + (a_2, 0) = (a_1 + a_2, a_1).$$

Thus, the equation gives

$$(a_1 + a_2, a_1) = (0,0),$$

so $a_1 + a_2 = 0$ and $a_1 = 0$. Plugging-in $a_1 = 0$ to $a_1 + a_2 = 0$, we see that $a_2 = 0$. Thus, we have shown that $a_1 = 0$ and $a_2 = 0$. □

(c) Are the vectors $(1,0,0)$, $(0,1,1)$ and $(1,1,1)$ linearly independent? Prove your answer.

Solution. No, they are not. To prove this, we need to find $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1(1,0,0) + a_2(0,1,1) + a_3(1,1,1) = (0,0,0)$$

and yet at least one of a_1, a_2, a_3 is nonzero. But this isn't hard as one can notice that the sum of the first two vectors equals the third vector, so

$$(1,0,0) + (0,1,1) - (1,1,1) = (0,0,0).$$

In other words, taking $a_1 = a_2 = 1$ and $a_3 = -1$ works. □

6. Consider the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = \frac{(-1)^n}{n^2}$. Determine whether the following are true or false, and prove your answer in either case.

(a) $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N\ |x_n| < \varepsilon$.

Solution. This is true and here is a proof. Fix arbitrary $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ (most likely in terms of ε) so that for any n bigger than N, $\frac{1}{n^2} < \varepsilon$.

On scratch paper (doesn't have to be included in your solution)**:** OK, we want $\frac{1}{n^2} < \varepsilon$. Now how large should n be for this to be true? Well, let's find out by solving the inequality $\frac{1}{n^2} < \varepsilon$ for n. Tu-tutu-tutu, we get $n > \frac{1}{\sqrt{\varepsilon}}$. Aha, so as long as n is bigger than $\frac{1}{\sqrt{\varepsilon}}$, I'd be fine. Wait, but this n is going to be $\ge N$, so if I choose my

N any natural number greater than $\frac{1}{\sqrt{\varepsilon}}$, this whole thing would work! For example, I can take $N = \lceil \frac{1}{\sqrt{\varepsilon}} \rceil + 1$. Oh boy, oh boy, why am I so clever?

On the official midterm paper (with a serious face)**:** We take $N = \lceil \frac{1}{\sqrt{\varepsilon}} \rceil + 1$ and let us check that $\forall n \geq N$ we indeed have $\frac{1}{n^2} < \varepsilon$. Fix arbitrary $n \geq N$. Thus, $n \geq \lceil \frac{1}{\sqrt{\varepsilon}} \rceil + 1$, so, in particular, $n > \frac{1}{\sqrt{\varepsilon}}$. Solving this inequality for ε gives $\frac{1}{n^2} < \varepsilon$. Have a pleasant day. □

(b) $\exists N \in \mathbb{N}\ \forall n \geq N\ \forall \varepsilon > 0\ |x_n| < \varepsilon$.

Solution. We show that this is false by proving its negation, which is:

$$\forall N \in \mathbb{N}\ \exists n \geq N\ \exists \varepsilon > 0\ \frac{1}{n^2} \geq \varepsilon.$$

Fix arbitrary $N \in \mathbb{N}$. We need to find $n \geq N$ and $\varepsilon > 0$ such that $\frac{1}{n^2} \geq \varepsilon$. But this is easy: take $n = N$ and $\varepsilon = \frac{1}{n^2}$. □